

# STRONG SPLITTER THEOREM

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**ABSTRACT.** The Splitter Theorem states that, if  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  such that, if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl, respectively, then there is a sequence  $M_0, \dots, M_n$  of 3-connected matroids with  $M_0 \cong N$ ,  $M_n = M$  and for  $i \in \{1, \dots, n\}$ ,  $M_i$  is a single-element extension or coextension of  $M_{i-1}$ . Observe that there is no condition on how many extensions may occur before a coextension must occur. In this paper, we give a strengthening of the Splitter Theorem, as a result of which we can obtain, up to isomorphism,  $M$  starting with  $N$  and at each step doing a 3-connected single-element extension or coextension, such that at most two consecutive single-element extensions occur in the sequence (unless the rank of the matroids involved are  $r(M)$ ). Moreover, if two consecutive single-element extensions by elements  $\{e, f\}$  are followed by a coextension by element  $g$ , then  $\{e, f, g\}$  form a triad in the resulting matroid. Using the Strong Splitter Theorem, we make progress toward the problem of determining the almost-regular matroids [6, 15.9.8]. *Find all 3-connected non-regular matroids such that, for all  $e$ , either  $M \setminus e$  or  $M/e$  is regular.* In [4] we determined the binary almost-regular matroids with at least one regular element (an element such that both  $M \setminus e$  and  $M/e$  is regular) by characterizing the class of binary almost-regular matroids with no minor isomorphic to one particular matroid that we called  $E_5$ . As a consequence of the Strong Splitter Theorem we can determine the class of binary matroids with an  $E_5$ -minor, but no  $E_4$ -minor.

## 1. Introduction

The matroid terminology follows Oxley [6]. Let  $M$  be a matroid and  $X$  be a subset of the ground set  $E$ . The *connectivity function*  $\lambda$  is defined as  $\lambda(X) = r(X) + r(E - X) - r(M)$ . Observe that  $\lambda(X) = \lambda(E - X)$ . For  $k \geq 1$ , a partition  $(A, B)$  of  $E$  is called a  $k$ -separation if  $|A| \geq k$ ,  $|B| \geq k$ , and  $\lambda(A) \leq k - 1$ . When  $\lambda(A) = k - 1$ , we call  $(A, B)$  an *exact  $k$ -separation*. When  $\lambda(A) = k - 1$  and  $|A| = k$  or  $|B| = k$ , we call  $(A, B)$  a *minimal exact  $k$ -separation*. For  $n \geq 2$ , we say  $M$  is  *$n$ -connected* if  $M$  has no  $k$ -separation for  $k \leq n - 1$ . A matroid is *internally  $n$ -connected* if it is  $n$ -connected and has no non-minimal exact  $n$ -separations. In particular, a simple matroid is 3-connected if  $\lambda(A) \geq 2$  for all partitions  $(A, B)$  with  $|A| \geq 3$  and  $|B| \geq 3$ . A 3-connected matroid is *internally 4-connected*

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if  $\lambda(A) \geq 3$  for all partitions  $(A, B)$  with  $|A| \geq 4$  and  $|B| \geq 4$ . To eliminate trivial cases, we shall also assume that a 3-connected matroid has at least four elements.

If  $M$  and  $N$  are matroids on the sets  $E$  and  $E \cup e$  where  $e \notin E$ , then  $M$  is a *single-element extension* of  $N$  if  $M \setminus e = N$ , and  $M$  is a *single-element coextension* of  $N$  if  $M^*$  is a single-element extension of  $N^*$ . If  $N$  is a 3-connected matroid, then an extension  $M$  of  $N$  is 3-connected provided  $e$  is not in a 1- or 2-element circuit of  $N$  and  $e$  is not a coloop of  $N$ . Likewise,  $M$  is a 3-connected coextension of  $N$  if  $M^*$  is a 3-connected extension of  $N^*$ .

In 1966 Tutte proved that for a 3-connected matroid  $M$  that is not a wheel or a whirl, there exists an element  $e \in E(M)$ , such that either  $M \setminus e$  or  $M/e$  is 3-connected [11]. In other words, if  $M$  is a 3-connected matroid, then  $M$  has a 3-connected proper minor  $M'$  such that  $|E(M) - E(M')| = 1$ , unless  $M$  is a wheel or whirl. In the case that  $M$  is a wheel or a whirl, for every  $e$  there is an  $f$  such that  $M \setminus e/f$  is 3-connected. So Tutte's theorem can be restated as follow: If  $M$  is a 3-connected matroid, then  $M$  has a 3-connected proper minor  $M'$  such that  $|E(M) - E(M')| \leq 2$ .

In 1972 Brylawski proved that if  $M$  is a 2-connected matroid with a proper 2-connected minor  $N$ , then there exists  $e \in E(M) - E(N)$  such that either  $M \setminus e$  or  $M/e$  is 2-connected and has  $N$  as a minor [2]. So when the matroid is 2-connected we can maintain 2-connectivity in  $M \setminus e$  or  $M/e$ , as well as the presence of a certain minor. In 1980 and 1981 Seymour and Tan independently proved that, if  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  such that if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl, respectively, then there exists  $e \in E(M) - E(N)$  such that either  $M \setminus e$  or  $M/e$  is 3-connected and has  $N$  as a minor [7, 8]. This is known as the Splitter Theorem. In other words,  $M$  has a 3-connected minor  $M'$  with  $|E(M) - E(M')| = 1$  and having an  $N$ -minor, unless  $M$  is a wheel or whirl, in which case  $M$  has a 3-connected proper minor  $M'$  with  $|E(M) - E(M')| = 2$  and having an  $N$ -minor. A formal statement of the Splitter Theorem appears below [6, 12.1.2].

**Theorem 1.1.** *Suppose  $N$  is a connected, simple, cosimple proper minor of a 3-connected matroid  $M$  such that, if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl-minor, respectively. Then  $M$  has a connected, simple, cosimple minor  $M'$  and an element  $e$  such that  $M' \setminus e$  or  $M'/e$  is isomorphic to  $N$ .  $\square$*

The next two results are reformulations of the Splitter Theorem and appear in [6, 12.1.3] and [6, 12.2.1]. Let  $\mathcal{M}$  be a class of matroids closed under minors and isomorphism. A *splitter*  $N$  for  $\mathcal{M}$  is a 3-connected matroid in  $\mathcal{M}$  such that no 3-connected matroid in  $\mathcal{M}$  has  $N$  as a proper minor.

**Corollary 1.2.** *Suppose  $N$  is a 3-connected matroid in  $\mathcal{M}$  such that, if  $N$  is a wheel or whirl then it is the largest wheel or whirl in  $\mathcal{M}$ . Suppose further that every 3-connected single-element extension and coextension of  $N$  does not belong to  $\mathcal{M}$ . Then  $N$  is a splitter for  $\mathcal{M}$ .  $\square$*

Checking if a matroid is a splitter is a potentially infinite task. The above reformulation of the Splitter Theorem turns this into a finite task, that can be easily checked.

Next, suppose  $\mathcal{M}$  is defined as having a specific 3-connected matroid in it, for example, the class of 3-connected binary matroids with an  $F_7$  or  $F_7^*$ -minor, but without an  $M(W_4)$ -minor. The third reformulation of the Splitter Theorem asserts that the entire class can be built up by performing single-element extensions and coextensions starting with the specified matroid and checking for the specified excluded minor(s).

**Corollary 1.3.** *Suppose  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  such that, if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl-minor, respectively. Then, there is a*

sequence  $M_0, \dots, M_n$  of 3-connected matroids with  $M_0 \cong N$ ,  $M_n = M$  and for  $i \in \{1, \dots, n\}$ ,  $M_i$  is a single-element extension or coextension of  $M_{i-1}$ .  $\square$

Subsequently, a couple of variations of the Splitter Theorem have been developed. For the variations of the Splitter Theorem, it suffices to state the results in terms of  $N$  being 3-connected since the general case when  $N$  is connected, simple, and cosimple is covered by the original Splitter Theorem. Observe that the minor  $M'$  is not required to have a single-element deletion or contraction equal to  $N$ , but only to have such a minor isomorphic to  $N$ . There is a counterexample to show the stronger statement of equality does not hold [6, 12.1 Ex. 7]. Truemper [6, 12.3.2] strengthened the conclusion by proving that, if  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$ , then  $M$  has a 3-connected minor  $M'$  and an element  $e$  such that  $co(M' \setminus e) = N$  or  $si(M'/e) = N$  and  $|E(M') - E(N)| \leq 3$ . Bixby and Coullard gave another similar variant [6, 12.3.6].

Coullard and Oxley [6, 12.3.1] showed that the restriction on excluding wheels and whirls can be weakened, so that instead of applying to all such matroids, it applies only to the smallest 3-connected wheels and whirls. They proved that if  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  that is not a wheel or a whirl and if  $N \cong W^2$ , then  $M$  has no  $W^3$ -minor and if  $N \cong M(W_3)$ , then  $M$  has no  $M(W_4)$ -minor, then  $M$  has a 3-connected minor  $M'$  and an element  $e$  such that  $M' \setminus e$  or  $M'/e$  is isomorphic to  $N$ .

We prove a new variant of the Splitter Theorem, the usefulness of which becomes apparent in the third section, where we prove structural results by applying it.

**Theorem 1.4.** *Suppose  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  such that, if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl-minor, respectively. Further, suppose  $m = r(M) - r(N)$ . Then there is a sequence of 3-connected matroids  $M_0, M_1, \dots, M_n$ , for some integer  $n \geq m$ , such that*

- (i)  $M_0 \cong N$ ;
- (ii)  $M_n = M$ ;
- (iii) for  $k \in \{1, 2, \dots, m\}$ ,  $r(M_k) - r(M_{k-1}) = 1$  and  $|E(M_k) - E(M_{k-1})| \leq 3$ ; and
- (iv) for  $m < k \leq n$ ,  $r(M_k) = r(M)$  and  $|E(M_k) - E(M_{k-1})| = 1$ .

Moreover, when  $|E(M_k) - E(M_{k-1})| = 3$ , for some  $1 \leq k \leq m$ ,  $E(M_k) - E(M_{k-1})$  is a triad of  $M_k$ .

Thus we can obtain, up to isomorphism,  $M$  starting with  $N$  and at each step doing a 3-connected single-element extension or coextension, such that at most two consecutive single-element extensions occur in the sequence (unless the rank of the matroids involved are  $r$ ). Moreover, if two consecutive single-element extensions by elements  $\{e, f\}$  are followed by a coextension by element  $g$ , then  $\{e, f, g\}$  form a triad in the resulting matroid. Finally, note that we can replace the restrictions on  $M$  and  $N$  by the weaker restrictions on  $M$  and  $N$  given by Coullard and Oxley.

## 2. PROOF OF THE STRONG SPLITTER THEOREM

In this section we give the proof of Theorem 1.4. Let us begin by proving a key lemma.

**Lemma 2.1.** *Suppose  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  and  $r(M) = r(N) + 1$ . Then, either*

- (i) *There is an element  $e \in E(M) - E(N)$  such that  $M \setminus e$  is 3-connected and  $N$  is a minor of  $M \setminus e$ ; or*
- (ii)  $|E(M) - E(N)| \leq 3$ .

Moreover, when  $|E(M) - E(N)| = 3$ ,  $T^* = E(M) - E(N)$  is triad of  $M$  and  $M \setminus T^* = N$ .

**Proof.** There is a set  $A$  of elements of  $M$  and an element  $b$  of  $M$  such that  $b \notin A$  and  $N = M \setminus A/b$ . If  $A = \emptyset$ , then (ii) follows. Assume that  $A \neq \emptyset$ . Choose  $e \in A$ . If  $M \setminus e$  is 3-connected, then (i) follows. Suppose that  $M \setminus e$  is not 3-connected. Let  $\{X, Y\}$  be a 2-separation for  $M \setminus e$ . As  $\{X \cap E(N), Y \cap E(N)\}$  is not a 2-separation for  $N$ , it follows that  $\min\{|X \cap E(N)|, |Y \cap E(N)|\} \leq 1$ , say  $|Y \cap E(N)| \leq 1$ . We do not lose generality by assuming that  $Y$  is closed in  $M$ . Using this 2-separation, we can decompose  $M \setminus e$  as the 2-sum of matroids  $M_X$  and  $M_Y$  such that  $E(M_X) = X \cup z$  and  $E(M_Y) = Y \cup z$ , for some  $z \notin E(M)$ . Observe that  $N$  is isomorphic to a minor of  $M_X$  because  $|E(N) - X| \leq 1$ . In particular,  $r(M_X) \geq r(N)$ . But

$$r(N) + 1 = r(M) = r(M \setminus e) = r(X) + r(Y) - 1 = r(M_X) + r(M_Y) - 1 \geq r(N) + 1$$

(As  $M \setminus e$  is simple, it follows that  $r(M_Y) = r(Y) \geq 2$ .) We must have equality along this display. Therefore

$$(1) \quad r(M_X) = r(N) \text{ and } r(M_Y) = 2.$$

In particular,

$$(2) \quad b \in Y.$$

We can be more precise about a minor  $N'$  of  $M_X$  isomorphic to  $N$ . Observe that  $N' = N = M_X \setminus [(A \cap X) \cup z]$ , when  $E(N) \cap Y = \emptyset$ , or  $N$  is obtained from  $N' = M_X \setminus (A \cap X)$  by relabeling  $z$  by  $z'$ , when  $E(N) \cap Y = \{z'\}$ . As  $Y$  is closed in  $M$ , it follows that  $M_X$  is simple because the series class of  $z$  in  $M_X$  is trivial. Observe that  $M_X$  is 3-connected because  $N'$  is a 3-connected restriction of  $M_X$  having the same rank as  $M_X$  and  $M_X$  is simple. If  $M_Y$  is not 3-connected, then  $M_Y$  is not simple because  $r(M_Y) = 2$ . In this case, there is an element  $f \in Y$  such that  $\{f, z\}$  is a parallel class of  $M_Y$  and  $M_Y \setminus f$  is 3-connected (recall that no two elements of  $Y$  are in parallel in  $M$  and so in  $M_Y$ ). We have two possibilities to consider:

- (a)  $M_Y$  is 3-connected. In this case,  $\{X, Y\}$  is the unique 2-separation for  $M \setminus e$ .
- (b)  $M_Y$  is not 3-connected. In this case,  $\{X, Y\}$  and  $\{X \cup f, Y - f\}$  are the 2-separations for  $M \setminus e$ .

Assume that  $g \in A \cap X$ . As  $N'$  and  $M_X$  are 3-connected matroid with the same rank and  $N'$  is a restriction of both  $M_X$  and  $M_X \setminus g$ , it follows that  $M_X \setminus g$  is 3-connected. Therefore  $\{X - g, Y\}$  is the unique 2-separation for  $M \setminus \{e, g\}$ , when (a) occurs, or  $\{X - g, Y\}$  and  $\{(X - g) \cup f, Y - f\}$  are the 2-separations for  $M \setminus \{e, g\}$ , when (b) occurs. Moreover,  $M \setminus \{e, g\}$  has no 1-separation. In both cases, each set in these 2-separations does not span  $e$  and so  $M \setminus g$  is 3-connected. We have (i) because  $N$  is a minor of  $M \setminus g$ . We may assume that

$$(3) \quad A \cap X = \emptyset.$$

With a similar argument, we conclude that  $M \setminus h$  is a 3-connected matroid having  $N$  as a minor, when  $h \in (Y - P) \cap A$ , provided  $|Y - P| \geq 3$ , where  $P$  is the parallel class of  $M_Y$  containing  $z$ . (Since at most one element of  $Y - P$  belongs to  $E(N)$  and at most one element of  $Y - P$  is equal to  $b$ , it follows that  $Y - P$  meets  $A$  provided  $|Y - P| \geq 3$ , that is, the element  $h$  exists.) Thus (i) also occurs unless

$$(4) \quad |Y - P| = 2.$$

We may assume the last identity otherwise the result follows. If  $|P| = 2$ , that is,  $P = \{z, f\}$ , then  $\{X, Y - f\}$  is the unique 2-separation for  $M \setminus \{e, f\}$ . Moreover,  $f \in A$  or  $f \in E(N)$ . If  $f \in A$ ,

$M \setminus f$  is a 3-connected matroid having  $N$  as a minor because  $e$  is not spanned by any set in this 2-separation. In this case, we have (i). Hence, we could also assume that

$$(5) \quad \text{if } |P| = 2, \text{ then } P - z \subseteq E(N).$$

By (2), (3) and (5),  $E(M) - E(N) \subseteq (Y - P) \cup e$ . By (4),  $|E(M) - E(N)| \leq 3$ . The first part of this result follows. Now, we establish the second part. Assume that

$$|E(M) - E(N)| = 3.$$

In this case, no element of  $Y - P$  belongs to  $N$ . So  $T^* = e \cup (Y - P) = E(M) - \text{cl}_M(X)$  is a triad of  $M$  avoiding  $E(N)$ . But  $b \in T^*$  and  $T^* - b = A$ . Therefore  $N = M \setminus T^*$ .  $\square$

**Proof of Theorem 1.4.** Suppose  $N$  is a 3-connected proper minor of a 3-connected matroid  $M$  such that, if  $N$  is a wheel or whirl then  $M$  has no larger wheel or whirl-minor, respectively. By Corollary 1.3, there is a sequence of 3-connected matroids  $N_0, N_1, \dots, N_n$  such that  $N_0 \cong N$ ,  $N_n = M$  and, for each  $i$  belonging to  $\{1, 2, \dots, n\}$ ,  $N_i$  is a single-element extension or coextension of  $N_{i-1}$ . Set  $m = r(M) - r(N)$ . There are indexes  $i_1, i_2, \dots, i_m$  such that  $0 < i_1 < i_2 < \dots < i_m$  and, for each  $k$  belonging to  $\{1, 2, \dots, m\}$ ,

$$r(N_{i_k}) - r(N_{i_{k-1}}) = 1$$

(That is,  $N_{i_k}$  is the first matroid in the sequence  $N_0, N_1, N_2, \dots, N_n$  having rank equal to  $r(N) + k$ .) Choose this sequence such that  $(i_1, i_2, \dots, i_m)$  is minimal in the lexicographic order. By Lemma 2.1,  $i_1 = |E(N_{i_1}) - E(N_0)| \leq 3$  and, for each  $k$  such that  $2 \leq k \leq m$ ,  $i_k - i_{k-1} = |E(N_{i_k}) - E(N_{i_{k-1}})| \leq 3$ . Moreover, when the equality holds, we have that  $E(N_{i_k}) - E(N_0)$  is a triad of  $E(N_3)$ , when  $k = 1$ , or  $E(N_{i_k}) - E(N_{i_{k-1}})$  is a triad of  $N_{i_k}$ , when  $k \geq 2$ .

The sequence of matroids that appear in the statement of the strong splitter theorem is

$$N_0, N_{i_1}, N_{i_2}, \dots, N_{i_m}, N_{i_m+1}, N_{i_m+2}, \dots, N_t$$

(That is, we remove all matroids having rank less than the rank of  $M$  except for those having the rank for the first time.)  $\square$

### 3. AN APPLICATION OF THE STRONG SPLITTER THEOREM

In this section we will use the Strong Splitter Theorem to make some progress on determining the class of almost-regular matroids. A non-graphic matroid  $M$  is *almost-graphic* if, for all elements  $e$ , either  $M \setminus e$  or  $M/e$  is graphic. A non-regular matroid is *almost-regular* if, for all elements  $e$ , either  $M \setminus e$  or  $M/e$  is regular. An element  $e$  for which both  $M \setminus e$  and  $M/e$  are regular is called a *regular element*. Determining these classes of matroids was listed as an unsolved problem in the first edition of Oxley's book *Matroid Theory*. In [4] we determined completely the class of almost-graphic matroids as well as the class of almost-regular matroids with at least one regular element. The problem that remains to be solved appears in the second edition as follows: [6, 15.9.8]: *Find all non-regular matroids  $M$  such that, for all elements  $e$ , exactly one of  $M \setminus e$  and  $M/e$  is regular.*

In order to determine the class of almost-regular matroids with at least one regular element, we turned the problem into a series of excluded-minor classes and determined the members in them. In the almost-graphic paper, we proved the following characterization of almost-regular matroids with no  $E_5$ -minor [4, 8.2].

**Theorem 3.1.** *Suppose  $M$  is a 3-connected binary almost-regular matroid with no  $E_5$ -minor. Then  $M \cong X_{12}$  or  $M$  or  $M^*$  is isomorphic to a 3-connected restriction of  $S_{3n+1}$  for  $n \geq 3$ ,  $\mathcal{F}_1(m, n, r)$  or  $\mathcal{F}_2(m, n, r)$  for  $m, n, r \geq 1$ .*

See [4] for a detailed description of the infinite families as well as the exceptional matroid  $X_{12}$  that is a splitter for the class. Matroids like  $T_{12}$  and  $X_{12}$  and the rank-5, 10-element matroids  $E_4$  and  $E_5$  (which are single-element coextensions of  $P_9$ ) play a useful role in the structure of binary matroids and feature in several papers [5]. Matrix representations for  $E_4$  and  $E_5$  are shown below.

$$E_4 = \left[ \begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 \end{array} \right] \quad E_5 = \left[ \begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array} \right]$$

To finish the almost-regular problem, one has to determine the almost-regular matroids with an  $E_5$ -minor. This is complicated by the presence of internally 4-connected members. However, in this paper we establish that the only thing left to do is to find the almost-regular matroids with both an  $E_5$  and an  $E_4$ -minor.

We found a rank-6, 12-element self-dual matroid that is a splitter for the class of binary 3-connected matroids with an  $E_5$ -minor and no  $E_4$ -minor. A matrix representations for  $M_{12}$  is shown below.

$$M_{12} = \left[ \begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_6 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

The main theorem in this section determines the class of matroids with an  $E_5$ -minor, but no  $E_4$ -minor. It has a finite list of members. With the exception of  $M_{12}$ , they are among the 3-connected restrictions of a rank-5, 17 element matroid, that we call  $R_{17}$ . The matroid  $R_{17}$  is an extension of both  $E_5$  and  $R_{10}$  (the unique splitter for regular matroids). Note that  $R_{17}$  has 3-connected restrictions that do not have an  $E_5$ -minor. A matrix representation for  $R_{17}$  is shown below.

$$R_{17} = \left[ \begin{array}{c|cccccccccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

The next result is the main theorem of this section.

**Theorem 3.2.** *Suppose  $M$  is a 3-connected binary matroid with an  $E_5$ -minor and no  $E_4$ -minor. Then  $M \cong M_{12}$  or  $M$  or  $M^*$  is isomorphic to  $R_{17}$  or is a 3-connected restrictions of  $R_{17}$  having an  $E_5$ -minor.*

**Proof.** The matroid  $E_5$  is self-dual and has seven non-isomorphic binary 3-connected single-element extensions, shown in Appendix Table A1. Observe that all the extensions, except  $A$ ,  $B$  and  $C$  have an  $E_4$ -minor. Matrix representations for  $A$ ,  $B$ , and  $C$  are given below.

$$A = \left[ \begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad B = \left[ \begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \quad C = \left[ \begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$



The proof is in three stages. First, we will show that all the coextensions of  $A$ ,  $B$ , and  $C$  have an  $E_4$ -minor with the exception of  $M_{12}$ . Suppose  $M$  is a coextension of  $A$ ,  $B$ ,  $C$ . Then a partial matrix representation for  $M$  is shown in Figure 1.

$$\left[ \begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ \hline & \hline \end{array} \right]$$

FIGURE 1. Structure of a coextension of  $A$ ,  $B$ ,  $C$

There are three types of rows that may be inserted into the last row on the right-hand side of the matrix in Figure 1.

- (i) rows that can be added to  $E_5$  to obtain a coextension with no  $E_4$ -minor, with a 0 or 1 as the last entry;
- (ii) the identity rows with a 1 in the last position;
- (iii) and the rows “in-series” to the right-hand side of matrices  $A$ ,  $B$ ,  $C$  with the last entry reversed.

Type I rows are  $[001110]$ ,  $[001111]$ ,  $[010010]$ ,  $[010011]$ ,  $[010100]$ ,  $[010101]$ ,  $[011000]$ ,  $[011001]$ ,  $[100110]$ ,  $[100111]$ ,  $[101010]$ ,  $[101011]$ ,  $[111010]$ , and  $[111011]$ . They are obtained from the Appendix Table A2. Type II rows are  $[100001]$ ,  $[010001]$ ,  $[001001]$ ,  $[000101]$ , and  $[000011]$ . Type III rows are specific to the matrices  $A$ ,  $B$ ,  $C$ . For matrix  $A$  they are  $[011111]$ ,  $[101101]$ ,  $[110110]$ ,  $[111101]$ ,  $[110000]$ . For matrix  $B$  they are  $[011110]$ ,  $[101101]$ ,  $[110111]$ ,  $[111100]$ , and  $[110000]$ . For  $C$  they are  $[011110]$ ,  $[101100]$ ,  $[110111]$ ,  $[111101]$ , and  $[110000]$ .

Most of the above rows result in matroids that have an  $E_4$ -minor. See red rows in the Appendix Table A3. Only a few coextensions must be specifically checked for an  $E_4$ -minor:  $(A, coextn11)$ ,  $(B, coextn8)$ ,  $(C, coextn8)$ ,  $(C, coextn9)$ ,  $(C, coextn10)$ ,  $(C, coextn12)$ , and  $(C, coextn14)$ .

Observe that,  $(A, coextn11)/11 \setminus 3 \cong E_4$ ,  $(C, coextn8)/12 \setminus 2 \cong E_4$ ,  $(C, coextn9)/12 \setminus 1 \cong E_4$ ,  $(C, coextn10)/12 \setminus 10 \cong E_4$ , and  $(C, coextn14)/12 \setminus 6 \cong E_4$ . Further, it is easy to check that  $(B, coextn8) \cong (C, coextn12)$  and this matroid does not have an  $E_4$ -minor. This is the matroid  $M_{12}$ .

Second, we must establish that  $M_{12}$  is a splitter for the class of matroids with an  $E_5$ -minor, but no  $E_4$ -minor. By Corollary 1.2 and the fact that  $M_{12}$  is self-dual, we only need to check the single-element coextensions of  $M_{12}$ . From Appendix Table A3 observe that  $M_{12}$ , as a coextension of  $C$ , may be obtained by adding exactly one row. Thus there are no further rows that may be added to form coextensions without an  $E_4$ -minor. It follows that  $M_{12}$  is a splitter for the class of binary matroids with an  $E_5$ , but no  $E_4$ -minor.

Third, we must show that if  $M$  has an  $E_5$  and no  $E_4$ -minor, then either  $M \cong M_{12}$  or  $r(M) \leq 5$ . To do this, let us begin by computing the single-element extensions of  $A$ ,  $B$ , and  $C$  with no  $E_4$ -minor. From Appendix Table A1, we may conclude that the only columns that can be added to  $E_5$  to obtain a matroid with no  $M^*(K_5 \setminus e)$ -minor are  $[00101]$ ,  $[00110]$ ,  $[01011]$ ,  $[01100]$ ,  $[10011]$ ,  $[11001]$ ,  $[11101]$ . Adding these columns gives us four non-isomorphic single-element extensions of  $A$ ,  $B$ , and  $C$ . They are  $D$ ,  $E$ ,  $F$ , and  $G$  shown below.

$$\begin{aligned}
D &= \left[ \begin{array}{c|cccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right] & E &= \left[ \begin{array}{c|cccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right] \\
F &= \left[ \begin{array}{c|cccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right] & G &= \left[ \begin{array}{c|cccccc} I_5 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right]
\end{aligned}$$

Suppose  $M$  is a coextension of  $D$ ,  $E$ ,  $F$ , or  $G$ . Then the structure of  $M$  is shown in Figure 2. Observe that one row and two columns may be added to  $E_5$ .

$$\left[ \begin{array}{c|cccccc} I_6 & 0 & 1 & 1 & 1 & 1 & & \\ & 1 & 0 & 1 & 1 & 0 & & \\ & 1 & 1 & 0 & 1 & 1 & & \\ & 1 & 1 & 1 & 1 & 0 & & \\ & 1 & 1 & 0 & 0 & 0 & & \\ & & & & & & & \end{array} \right]$$

FIGURE 2. Structure of a coextension of  $D$ ,  $E$ ,  $F$ ,  $G$

Once again three types of rows may be added.

- (i) the rows that can be added to  $D$ ,  $E$ ,  $F$ , or  $G$  to obtain a coextension with no  $E_4$ -minor, with a 0 or 1 in the last entry.
- (ii) the identity rows with a 1 in the last position;
- (iii) and the rows “in-series” to the right-hand side of the matrices with the last entry reversed.

Suppose  $M$  is the coextension obtained by adding a Type I row, then  $M \setminus 13$  is 3-connected. However, the only rank-6, 12-element matroid in the class is  $M_{12}$  and it is a splitter; a contradiction. Thus we may assume  $M \setminus 13$  is not 3-connected.

Adding a Type II or III row (with the exception of  $[0000011]$ ) causes  $M \setminus e$  to be 3-connected where  $e \in \{12, 13\}$  (and again there are no such matroids except  $M_{12}$  which is a splitter). So the only coextension we must check is the one formed by adding row  $[0000011]$ . That is the coextension in which  $\{6, 11, 12\}$  is a triad. Let  $D'$ ,  $E'$ ,  $F'$ , and  $G'$  be the coextensions of  $D$ ,  $E$ ,  $F$ , and  $G$ , respectively, obtained by coextending by row  $[0000011]$ . Then in each case we can find an  $E_4$  minor. In particular,  $D'/1 \setminus \{3, 11\} \cong E_4$ ,  $E'/1 \setminus \{7, 11\} \cong E_4$ ,  $F'/1 \setminus \{7, 11\} \cong E_4$ , and  $G'/1 \setminus \{7, 11\} \cong E_4$ .

Finally, observe that if  $M$  is an extension of  $E_5$  of size  $k \geq 13$ , then for some  $e \in \{11, \dots, k\}$ ,  $M \setminus e$  is 3-connected. The result follows from Theorem 1.4.  $\square$

The next theorem uses Theorem 3.2 to characterize the almost-regular matroids with an  $E_5$ -minor, but no  $E_4$ -minor.

**Theorem 3.3.** *Suppose  $M$  is a 3-connected binary almost-regular matroid with an  $E_5$ -minor. Then, either  $M$  has an  $E_4$ -minor or  $M \cong E_5$ ,  $B$ , or  $B^*$ .*

**Proof.** We begin by looking at the single-element extensions of  $E_5$  and determining that  $B$  and  $H$  are the only ones that are almost-regular. Note that  $B$  and  $H$  are  $E_{5,11}$  and  $D_{5,11}$  in Appendix Table III in [4]. Further, note that  $H$  has an  $E_4$ -minor. Since  $B$  is formed by adding just one column to  $E_5$ , no further extension of  $E_5$  is almost-regular without an  $E_4$ -minor. The result then follows from Theorem 3.2 because  $M_{12}$  is not almost-regular.  $\square$



Thus we may assume an almost-regular matroid with an  $E_5$ -minor must also have an  $E_4$ -minor (with the exception of  $B$  and  $B^*$ ) and more importantly, it must have an  $H$ -minor, where  $H$  is the matroid shown below:

$$H = \left[ \begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

As a consequence of the above results we can strengthen Theorem 3.1 as follows:

**Theorem 3.5.** *Suppose  $M$  is a 3-connected binary almost-regular matroid with no  $H$  or  $H^*$ -minor. Then  $M \cong E_5$ ,  $B$ ,  $B^*$ ,  $X_{12}$ , or  $M$  or  $M^*$  is isomorphic to a 3-connected restriction of  $S_{3n+1}$  for  $n \geq 3$ ,  $\mathcal{F}_1(m, n, r)$  or  $\mathcal{F}_2(m, n, r)$  for  $m, n, r \geq 1$ .*

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## Appendix

Extension Columns				Name	$E_4$ -minor
[00101]	[00110]	[01011]	[01100]	A	No
[10011]				B	No
[11001]	[11101]			C	No
[00011]	[00111]	[01001]	[01101]		Yes
[01010]	[01110]				Yes
[10001]	[10010]	[11011]	[11100]		Yes
[10101]	[10110]	[11000]	[11111]	H	Yes

Table A1: Single-element extensions of  $E_5$

Coextension Rows				Name
[00111]	[01001]	[01010]	[01100]	$A^*$
[10011]				$B^*$
[10101]	[11101]			$C^*$
[00011]	[00101]	[01011]	[01101]	
[00110]	[01110]			
[10001]	[10010]	[10111]	[11100]	
[10100]	[11001]	[11010]	[11111]	$H^*$

Table A2: Single-element coextensions of  $E_5$ 

Matroid	Name	Coextension Row
$A$	coext 1	[000011] [000101] [001010] [011010] [101111] [111001]
	coext 2	[000110] [110011] [110101]
	coext 3	[000111] [101011] [111011]
	coext 4	[001001] [010110] [011111]
	coext 5	[001011] [011011] [100111]
	coext 6	[001100] [011100] [110000]
	coext 7	[001101] [010010] [010100] [011101] [101110] [111000]
	coext 8	[001110] [011000] [101101] [110010] [110100] [111101]
	coext 9	[001111] [011001] [100011] [100101] [101010] [111010]
	coext 10	[010001] [100010] [100100]
	coext 11	[010011] [010101] [100110]
	coext 12	[010111]
	coext 13	[100001] [101000] [111110]
	coext 14	[101001] [110110] [111111]
$B$	coext 1	[000011] [000101] [000110] [001001] [001010] [001111] [010010] [010100] [010111] [011000] [011011] [011110]
	coext 2	[000111] [001011] [010110] [011010]
	coext 3	[001100] [010001] [011101]
	coext 4	[001101] [001110] [010011] [010101] [011001] [011100]
	coext 5	[100001] [100010] [100100] [101000] [101101] [101110] [110000] [110011] [110101] [111001] [111100] [111111]
	coext 6	[100011] [100101] [101010] [101111] [111000] [111011]
	coext 7	[100110] [101001] [110010] [110100] [110111] [111110]
	coext 8	[100111] [101011] [111010]
$C$	coext 1	[000011] [000101] [001001] [001111] [010010] [010100] [011000] [011110] [100010] [100100] [101000] [101110] [110011] [110101] [111001] [111111]
	coext 2	[000110] [010111]
	coext 3	[000111] [010110] [100110] [110111]
	coext 4	[001010] [011011]
	coext 5	[001011] [011010] [101010] [111011]
	coext 6	[001100] [011101]
	coext 7	[001101] [011100] [101100] [111101]
	coext 8	[001110] [010011] [010101] [011001]
	coext 9	[010001]
	coext 10	[100001] [110000]
	coext 11	[100011] [100101] [101111] [111000]
	coext 12	[100111]
	coext 13	[101001] [110010] [110100] [111110]
	coext 14	[101011] [111010]

Table A3: Single-element coextensions of  $A$   $B$  and  $C$